Math 347 Worksheet

Worksheet 12: divisibility properties

November 12, 2018

1) Use induction on $a + b$ to prove that:

If a and b are relatively prime integers, then there exist m and n integers such that

$$ma + nb = 1.$$

2) Use 1) to prove that

If a and b are relatively prime and a divides qb, then a divides q.

3) Use 2) to prove that

(i) If a prime p divides a product of two integers, then p divides one of them;

(ii) If a prime p divides a product of k integers, then p divides one of them;

4) Let $a, b \in \mathbb{Z}$.

(i) Prove that $\gcd(a + b, b) = \gcd(a, b)$;

(ii) Prove that $\gcd(a + b, a - b) = \gcd(2a, a - b) = \gcd(a + b, 2b)$;

(iii) Generalize your proof of 1) to prove that for any integer number k

$$\gcd(a, b) = \gcd(a - kb, b).$$

5) (Uniqueness of division). If $a, b \in \mathbb{Z}$, prove that there exists exactly one pair of numbers $k, r \in \mathbb{Z}$ such that

(a) $0 \leq r < |b| - 1$, and;

(b) $a = kb + r$.

6) Given two integer numbers a, n

we write $a|n$ to mean "a divides n".

Prove that if $\gcd(a, b) = 1$ and $a|n$ and $b|n$, then $ab|n$. Give an example where this fails if $\gcd(a, b) \neq 1$.